Worksheet 4.7 Polynomials

Section 1 Introduction to Polynomials

A polynomial is an expression of the form

$$p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n \quad (n \in \mathbb{N})$$

where $p_0, p_1, \ldots, p_n$ are constants and x is a variable.

Example 1 : $f(x) = 3 + 5x + 7x^2 - x^4 \quad (p_0 = 3,\ p_1 = 5,\ p_2 = 7,\ p_3 = 0,\ p_4 = -1)$

Example 2 : $g(x) = 2x^3 + 3x \quad (p_0 = 0,\ p_1 = 3,\ p_2 = 0,\ p_3 = 2)$

- The constants $p_0, \ldots, p_n$ are called coefficients. In Example 1 the coefficient of x is 5 and the coefficient of x^4 is -1. The term which is independent of x is called the constant term. In Example 1 the constant term of $f(x)$ is 3; in Example 2 the constant term of $g(x)$ is 0.
- A polynomial $p_0 + p_1 x + \cdots + p_n x^n$ is said to have degree n, denoted $\deg n$, if $p_n \neq 0$ and x^n is the highest power of x which appears. In Example 1 the degree of $f(x)$ is 4; in Example 2 the degree of $g(x)$ is 3.
- A zero polynomial is a polynomial whose coefficients are all 0 i.e. $p_0 = p_1 = \cdots = p_n = 0$.
- Two polynomials are equal if all the coefficients of the corresponding powers of x are equal.

Exercises:

1. Find (i) the constant term, (ii) the coefficient of x^4 and (iii) the degree of the following polynomials.

   (a) $x^4 + x^3 + x^2 + x + 1$

   (b) $9 - 3x^2 + 7x^3$

   (c) $x - 2$

   (d) $10x^5 - 3x^4 + 5x + 6$

(e) $3x^6 + 7x^4 + 2x$

(f) $7 + 5x + x^2 - 6x^4$

2. Suppose $f(x) = 2ax^3 - 3x^2 - b^2x - 7$ and $g(x) = cx^4 + 10x^3 - (d+1)x^2 - 4x + e$. Find values for constants a, b, c, d and e given that $f(x) = g(x)$.

Section 2 Operation on Polynomials

Suppose $h(x) = 3x^2 + 4x + 5$
$k(x) = 7x^2 - 3x$

- Addition/Subtraction: To add/subtract polynomials we combine like terms.

  Example 1 : We have

$$h(x) + k(x) = 10x^2 + x + 5$$

- Multiplication: To multiply polynomials we expand and then simplify their product.

  Example 2 : We have

$$\begin{aligned} h(x) \cdot k(x) &= (3x^2 + 4x + 5)(7x^2 - 3x) \\ &= 21x^4 - 9x^3 + 28x^3 - 12x^2 + 35x^2 - 15x \\ &= 21x^4 + 19x^3 + 23x^2 - 15x \end{aligned}$$

- Substitution: We can substitute in different values of x to find the value of our polynomial at this point.

  Example 3 : We have

$$\begin{aligned} h(1) &= 3(1)^2 + 4(1) + 5 = 12 \\ k(-2) &= 7(-2)^2 - 3(-2) = 34 \end{aligned}$$

- Division: To divide one polynomial by another we use the method of long division.

  Example 4 : Suppose we wanted to divide $3x^3 - 2x^2 + 4x + 7$ by $x^2 + 2x$.

$$\begin{array}{r|l} & 3x - 8 \\ \hline x^2 + 2x & 3x^3 - 2x^2 + 4x + 7 \\ & \underline{3x^3 + 6x^2} \\ & \quad -8x^2 + 4x \\ & \quad \underline{-8x^2 - 16x} \\ & \qquad\qquad 20x + 7 \end{array}$$

So $3x^3 - 2x^2 + 4x + 7$ divided by $x^2 + 2x$ gives us a quotient of $3x - 8$ with a remainder of $20x + 7$. We have

$$3x^3 - 2x^2 + 4x + 7 = (x^2 + 2x)(3x - 8) + (20x + 7).$$

More formally, suppose $p(x)$ and $f(x)$ are polynomials where $\deg p(x) \geq \deg f(x)$. Then dividing $p(x)$ by $f(x)$ gives us the identity

$$p(x) = f(x)q(x) + r(x),$$

where $q(x)$ is the quotient, $r(x)$ is the remainder and $\deg r(x) < \deg f(x)$.

<u>Example 5</u> : Dividing $p(x) = x^3 - 7x^2 + 4$ by $f(x) = x - 1$ we obtain the following result:

$$\begin{array}{r|l} & x^2 - 6x - 6 \\ \cline{2-2} x - 1 & x^3 - 7x^2 + 0x + 4 \\ & \underline{x^3 - x^2} \\ & \quad -6x^2 + 0x \\ & \quad \underline{-6x^2 + 6x} \\ & \qquad\qquad -6x + 4 \\ & \qquad\qquad \underline{-6x + 6} \\ & \qquad\qquad\qquad -2 \end{array}$$

Here the quotient is $q(x) = x^2 - 6x - 6$ and the remainder is $r = -2$. <u>Note</u>: As we can see, division doesn't always produce a polynomial annswer- sometimes there's just a constant remainder.

Exercises:

1. Perform the following operations and find the degree of the result.
   (a) $(2x - 4x^2 + 7) + (3x^2 - 12x - 7)$
   (b) $(x^2 + 3x)(4x^3 - 3x - 1)$
   (c) $(x^2 + 2x + 1)^2$
   (d) $(5x^4 - 7x^3 + 2x + 1) - (6x^4 + 8x^3 - 2x - 3)$
2. Let $p(x) = 3x^4 + 7x^2 - 10x + 4$. Find $p(1)$, $p(0)$ and $p(-2)$.
3. Carry out of the following divisions and write your answer in the form $p(x) = f(x)q(x) + r(x)$.
   (a) $(3x^3 - x^2 + 4x + 7) \div (x + 2)$
   (b) $(3x^3 - x^2 + 4x + 7) \div (x^2 + 2)$

(c) $(x^4 - 3x^2 - 2x + 4) \div (x - 1)$
(d) $(5x^4 + 30x^3 - 6x^2 + 8x) \div (x^2 - 3x + 1)$
(e) $(3x^4 + x) \div (x^2 + 4x)$

4. Find the quotient and remainder of the following divisions.

   (a) $(2x^4 - 2x^2 - 1) \div (2x^3 - x - 1)$
   (b) $(x^3 + 2x^2 - 5x - 3) \div (x + 1)(x - 2)$
   (c) $(5x^4 - 3x^2 + 2x + 1) \div (x^2 - 2)$
   (d) $(x^4 - x^2 - x) \div (x + 2)^2$
   (e) $(x^4 + 1) \div (x + 1)$

Section 3 Remainder Theorem

We have seein in Section 2 that if a polynomial $p(x)$ is divided by polynomial $f(x)$, where $\deg p(x) \geq \deg f(x)$, we obtain the expression $p(x) = f(x)q(x) + r(x)$, where $q(x)$ is the quotient, $r(x)$ is the remainder and $\deg r(x) < \deg f(x)$, or $r = 0$.

Now suppose $f(x) = x - a$, where $a \in \mathbb{R}$, then

$$\begin{array}{rrcl} & p(x) & = & (x-a)q(x) + r(x) \\ \text{i.e} & p(x) & = & (x-a)q(x) + r, \quad \text{since } \deg r < \deg f. \end{array}$$

If we let $x = a$ then we get

$$\begin{array}{rrcl} & p(a) & = & (a-a)q(a) + r \\ \text{i.e.} & p(a) & = & r. \end{array}$$

So the remainder when $p(x)$ is divided by $x - a$ is $p(a)$. This important result is known as the remainder theorem

<u>Remainder Theorem</u>: If a polynomial $p(x)$ is divided by $(x - a)$, then the remainder is $p(a)$.

<u>Example 1</u> : Find the remainder when $x^3 - 7x^2 + 4$ is divided by $x - 1$.

Instead of going through the long division process to find the remainder, we can now use the remainder theorem. The remainder when $p(x) = x^3 - 7x^2 + 4$ is divided by $x - 1$ is

$$p(1) = (1)^3 - 7(1)^2 + 4 = -2.$$

Note: Checking this using long division will give the same remainder of -2 (see Example 5 from Section 2).

Exercises:

1. Using the remainder theorem find the remainder of the following divisions and then check your answers by long divsion.
   (a) $(4x^3 - x^2 + 2x + 1) \div (x - 5)$
   (b) $(3x^2 + 12x + 1) \div (x - 1)$
2. Using the remainder theorem find the remainder of the following divisions.
   (a) $(x^3 - 5x + 6) \div (x - 3)$
   (b) $(3x^4 - 5x^2 - 20x - 8) \div (x + 1)$
   (c) $(x^4 - 7x^3 + x^2 - x - 1) \div (x + 2)$
   (d) $(2x^3 - 2x^2 + 3x - 2) \div (x - 2)$

Section 4 The Factor Theorem and Roots of Polynomials

The remainder theorem told us that if $p(x)$ is divided by $(x - a)$ then the remainder is $p(a)$. Notice theat the remainder $p(a) = 0$ then $(x - a)$ fully divides into $p(x)$ i.e. $(x - a)$ is a factor of $p(x)$. This is the factor theorem.

Factor Theorem: Suppose $p(x)$ is a polynomial and $p(a) = 0$. Then $(x - a)$ is a factor of $p(x)$ and we can write $p(x) = (x - a)q(x)$ for some polynomial $q(x)$.

Note: If $p(a) = 0$ we call $x = a$ a root of $p(x)$.

We can use trial and error to find solutions of polynomial $p(x)$ by finding a number a whre $p(a) = 0$. If we can find such a number a then we know $(x - a)$ is a factor of $p(x)$, and then we can use long division to find the remaining factors of $p(x)$.

Example 1 :

a) Find all the factors of $p(x) = 6x^3 - 17x^2 + 11x - 2$.
b) Hence find all the solutions to $6x^3 - 17x^2 + 11x - 2 = 0$.

Solution a) By trial and error notice that

$p(2) = 48 - 66 + 22 - 2 = 0$
i.e. 2 is a root of $p(x)$.
So $x - 2$ is a factor of $p(x)$.

To find other factors we'll divide $p(x)$ by $(x - 2)$.

$$
\begin{array}{r|l}
 & 6x^2 - 5x + 1 \\
\hline
x - 2 & 6x^3 - 17x^2 + 11x - 2 \\
 & \underline{6x^3 - 12x^2} \\
 & \qquad -5x^2 + 11x \\
 & \qquad \underline{-5x^2 + 10x} \\
 & \qquad\qquad\qquad x - 2 \\
 & \qquad\qquad\qquad \underline{x - 2} \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

So $p(x) = (x - 2)(6x^2 - 5x + 1)$. Now notice

$$
\begin{aligned}
6x^2 - 5x + 1 &= 6x^2 - 3x - 2x + 1 \\
&= 3x(2x - 1) - (2x - 1) \\
&= (3x - 1)(2x - 1)
\end{aligned}
$$

So $p(x) = (x-2)(3x-1)(2x-1)$ and its factors are $(x-2)$, $(3x-1)$ and $(2x-1)$.

Solution b) The solutions to $p(x) = 0$ occur when

$$x - 2 = 0, \qquad 3x - 1 = 0, \qquad 2x - 1 = 0.$$

That is,

$$x = 2, \qquad x = \frac{1}{3}, \qquad x = \frac{1}{2}.$$

Exercises:

1. For each of the following polynomials find (i) its factors; (ii) its roots.
   (a) $x^3 - 3x^2 + 5x - 6$
   (b) $x^3 + 3x^2 - 9x + 5$
   (c) $6x^3 - x^2 - 2x$
   (d) $4x^3 - 7x^2 - 14x - 3$
2. Given that $x - 2$ is a factor of the polynomial $x^3 - kx^2 - 24x + 28$, find k and the roots of this polynomial.

3. Find the quadratic whose roots are -1 and $\frac{1}{3}$ and whose value at $x = 2$ is 10.

4. Find the polynomial of degree 3 which has a root at -1, a double root at 3 and whose value at $x = 2$ is 12.

5. (a) Explain why the polynomial $p(x) = 3x^2 + 11x^2 + 8x - 4$ has at least 1 root in the interval from $x = 0$ to $x = 1$.

   (b) Find all the roots of this polynomial.

Exercises 4.7 Polynomials

1. Find the quotient and remainder of the following divisions.
   (a) $(x^3 - x^2 + 8x - 5) \div (x^2 - 7)$
   (b) $(x^3 + 5x^2 + 15) \div (x + 3)$
   (c) $(2x^3 - 6x^2 - x + 6) \div (x - 6)$
   (d) $(x^4 + 3x^3 - x^2 - 2x - 7) \div (x^2 + 3x + 1)$

2. Find the factors of the following polynomials.
   (a) $3x^3 - 8x^2 - 5x + 6$
   (b) $x^3 - 4x^2 + 6x - 2$
   (c) $2x^3 + 5x^2 - 3x$
   (d) $x^3 + 6x^2 + 12x + 8$

3. Solve the following equations.
   (a) $x^3 - 3x^2 + x + 2 = 0$
   (b) $5x^3 + 23x^2 + 10x - 8 = 0$
   (c) $x^3 - 8x^2 + 21x - 18 = 0$
   (d) $x^3 - 2x^2 + 5x - 4 = 0$
   (e) $x^3 + 5x^2 - 4x + 20 = 0$

4. Consider the polynomial $p(x) = x^3 - 4x^2 + ax - 3$.
   (a) Find a if, when $p(x)$ is divided by $x + 1$, the remainder is -12.
   (b) Find all the factors of $p(x)$.

5. Consider the polynomial $h(x) = 3x^3 - kx^2 - 6x + 8$.
   (a) Given that $x - 4$ is a factor of $h(x)$, find k and find the other factors of $h(x)$.
   (b) Hence find all the roots of $h(x)$.

6. Find the quadratic whose roots are -3 and $\frac{1}{5}$ and whose value at $x = 0$ is -3.

7. Find the quadratic which has a remainder of -6 when divided by $x - 1$, a remainder of -4 when divided by $x - 3$ and no remainder when divided by $x + 1$.

8. Find the polynomial of degree 3 which has roots at $x = 1$, $x = 1 + \sqrt{2}$ and $x = 1 - \sqrt{2}$, and whose value at $x = 2$ is -2.

Answers 4.7

Section 1

1. (a) (i) 1 (ii) 1 (iii) 4
   (b) (i) 9 (ii) 0 (iii) 3
   (c) (i) -2 (ii) 0 (iii) 1
   (d) (i) 6 (ii) -3 (iii) 5
   (e) (i) 0 (ii) 7 (iii) 6
   (f) (i) 7 (ii) -6 (iii) 4

2. $a = 5$, $b = \pm 2$, $c = 0$, $d = 2$ and $e = -7$.

Section 2

1. (a) $-x^2 - 10x, \quad \deg = 2$
   (b) $4x^5 + 12x^4 - 3x^3 - 10x^2 - 3x, \quad \deg = 5$
   (c) $x^4 + 4x^3 + 6x^2 + 4x + 1, \quad \deg = 4$
   (d) $-x^4 - 15x^3 + 6x + 4, \quad \deg = 4$

2. $p(1) = 4, \quad p(0) = 4, \quad p(-2) = 100$

3. (a) $3x^3 - x^2 + 4x + 7 = (x + 2)(3x^2 - 7x + 18) - 29$
   (b) $3x^3 - x^2 + 4x + 7 = (x^2 + 2)(3x - 1) + (-2x + 9)$
   (c) $x^4 - 3x^2 + 2x + 4 = (x - 1)(x^3 + x^2 - 2x) + 4$
   (d) $5x^4 + 30x^3 - 6x^2 + 8x = (x^2 - 3x + 1)(5x^2 + 45x + 64) + (155x - 64)$
   (e) $3x^4 + x = (x^2 + 4x)(3x^2 - 12x + 48) - 191x$

4. (a) $q(x) = x, \quad r(x) = -x^2 + x - 1$
   (b) $q(x) = x + 3, \quad r(x) = 3$
   (c) $q(x) = 5x^2 + 7, \quad r(x) = 2x + 15$
   (d) $q(x) = x^2 - 4x + 15, \quad r(x) = 75x + 60$
   (e) $q(x) = x^3 - x^2 + x - 1, \quad r(x) = 2$

Section 3

1. (a) 486 (b) 16

2. (a) 18 (b) 10 (c) 77 (d) 12

Section 4

1. (a) $(x-2)(x^2-x+3)$ (c) $x(2x+1)(3x-2)$
   (b) $(x-1)^2(x+5)$ (d) $(4x+1)(x-3)(x+1)$

2. $k=3$, roots: 2 (double), -7

3. $6x^2+4x-2$

4. $4x^3-20x^2+12x+36$

5. (b) $\frac{1}{3}$, -2

Exercises 4.7

1. (a) $q(x)=x-1$, $r(x)=x+2$ (c) $q(x)=2x^2-1$, $r=0$
   (b) $q(x)=x^2-2x-6$, $r=33$ (d) $q(x)=x-1$, $r(x)=4x-5$

2. (a) $(3x-2)(x+1)(x-3)$
   (b) $(x-2)(x^2-2x+2)$
   (c) $x(2x-1)(x+3)$
   (d) $(x-1)(x^2+x-2)$
   (e) $(x+2)^3$

3. (a) 2, $\frac{-1+\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$
   (b) $\frac{2}{5}$, -1, -4
   (c) 3, 2
   (d) 1
   (e) 2, -2, -5

4. (a) $a=5$
   (b) factors are $(x-3)$ and (x^2-x+1)

5. (a) $k = 11$, factors are $(x-4)$, $(x+1)$ and $(3x-2)$
   (b) roots are 4, -1 and $\frac{2}{3}$
6. $5x^2 + 14x - 3$
7. $x^2 - 3x - 4$
8. $2x^3 - 6x^2 + 2x + 2$