

FORM TP 2013233



TEST CODE 02134020

MAY/JUNE 2013

CARIBBEAN EXAMINATIONS COUNCIL

CARIBBEAN ADVANCED PROFICIENCY EXAMINATION®

PURE MATHEMATICS

UNIT 1 – Paper 02

ALGEBRA, GEOMETRY AND CALCULUS

2 hours 30 minutes

14 MAY 2013 (p.m.)

This examination paper consists of **THREE** sections: Module 1, Module 2 and Module 3.

Each section consists of 2 questions.

The maximum mark for each Module is 50.

The maximum mark for this examination is 150.

This examination consists of 6 printed pages.

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. **DO NOT** open this examination paper until instructed to do so.
2. Answer **ALL** questions from the **THREE** sections.
3. Write your solutions, with full working, in the answer booklet provided.
4. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to three significant figures.

Examination Materials Permitted

Graph paper (provided)

Mathematical formulae and tables (provided) – Revised 2012

Mathematical instruments

Silent, non-programmable, electronic calculator

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02134020/CAPE 2013



SECTION A (Module 1)

Answer BOTH questions.

1. (a) Let p and q be two propositions. Construct a truth table for the statements
- (i) $p \rightarrow q$ [1 mark]
 - (ii) $\sim(p \wedge q)$. [2 marks]
- (b) A binary operator \oplus is defined on a set of positive real numbers by
- $$y \oplus x = y^2 + x^2 + 2y + x - 5xy.$$
- Solve the equation $2 \oplus x = 0$. [5 marks]
- (c) Use mathematical induction to prove that $5^n + 3$ is divisible by 2 for all values of $n \in \mathbb{N}$. [8 marks]
- (d) Let $f(x) = x^3 - 9x^2 + px + 16$.
- (i) Given that $(x + 1)$ is a factor of $f(x)$, show that $p = 6$. [2 marks]
 - (ii) Factorise $f(x)$ completely. [4 marks]
 - (iii) Hence, or otherwise, solve $f(x) = 0$. [3 marks]

Total 25 marks

2. (a) Let $A = \{x : x \in \mathbb{R}, x \geq 1\}$.
A function $f: A \rightarrow \mathbb{R}$ is defined as $f(x) = x^2 - x$. Show that f is one to one. [7 marks]
- (b) Let $f(x) = 3x + 2$ and $g(x) = e^{2x}$.
- (i) Find
- a) $f^{-1}(x)$ and $g^{-1}(x)$ [4 marks]
- b) $f[g(x)]$ (or $f \circ g(x)$). [1 mark]
- (ii) Show that $(f \circ g)^{-1}(x) = g^{-1}(x) \circ f^{-1}(x)$. [5 marks]
- (c) Solve the following:
- (i) $3x^2 + 4x + 1 \leq 5$ [4 marks]
- (ii) $|x + 2| = 3x + 5$ [4 marks]

Total 25 marks

SECTION B (Module 2)

Answer BOTH questions.

3. (a) (i) Show that $\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$. [4 marks]
- (ii) Hence, or otherwise, solve $\sin 2\theta - \tan \theta = 0$ for $0 \leq \theta \leq 2\pi$. [8 marks]
- (b) (i) Express $f(\theta) = 3 \cos \theta - 4 \sin \theta$ in the form $r \cos(\theta + \alpha)$ where $r > 0$ and $0^\circ \leq \alpha \leq \frac{\pi}{2}$. [4 marks]
- (ii) Hence, find
- a) the maximum value of $f(\theta)$ [2 marks]
- b) the minimum value of $\frac{1}{8 + f(\theta)}$. [2 marks]
- (iii) Given that the sum of the angles A , B and C of a triangle is π radians, show that
- a) $\sin A = \sin(B + C)$ [3 marks]
- b) $\sin A + \sin B + \sin C = \sin(A + B) + \sin(B + C) + \sin(A + C)$. [2 marks]

Total 25 marks

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4. (a) A circle C is defined by the equation $x^2 + y^2 - 6x - 4y + 4 = 0$.
- (i) Show that the centre and the radius of the circle, C , are $(3, 2)$ and 3 , respectively. [3 marks]
- (ii) a) Find the equation of the normal to the circle C at the point $(6, 2)$. [3 marks]
- b) Show that the tangent to the circle at the point $(6, 2)$ is parallel to the y -axis. [3 marks]
- (b) Show that the Cartesian equation of the curve that has the parametric equations
- $$x = t^2 + t, \quad y = 2t - 4$$
- is $4x = y^2 + 10y + 24$. [4 marks]
- (c) The points $A(3, -1, 2)$, $B(1, 2, -4)$ and $C(-1, 1, -2)$ are three vertices of a parallelogram $ABCD$.
- (i) Express the vectors \vec{AB} and \vec{BC} in the form $xi + yj + zk$. [3 marks]
- (ii) Show that the vector $\mathbf{r} = -16\mathbf{j} - 8\mathbf{k}$ is perpendicular to the plane through A , B and C . [5 marks]
- (iii) Hence, find the Cartesian equation of the plane through A , B and C . [4 marks]

Total 25 marks

SECTION C (Module 3)

Answer BOTH questions.

5. (a) A function $f(x)$ is defined as $f(x) = \begin{cases} x + 2, & x < 2 \\ x^2, & x > 2 \end{cases}$.

(i) Find $\lim_{x \rightarrow 2} f(x)$. [4 marks]

(ii) Determine whether $f(x)$ is continuous at $x = 2$. Give a reason for your answer. [2 marks]

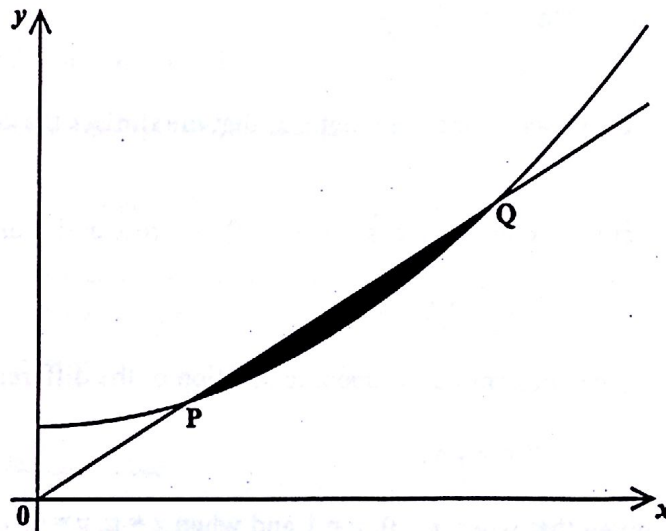
(b) Let $y = \frac{x^2 + 2x + 3}{(x^2 + 2)^3}$. Show that $\frac{dy}{dx} = \frac{-4x^3 - 10x^2 - 14x + 4}{(x^2 + 2)^4}$. [5 marks]

(c) The equation of an ellipse is given by

$$x = 1 - 3 \cos \theta, \quad y = 2 \sin \theta, \quad 0 \leq \theta \leq 2\pi.$$

Find $\frac{dy}{dx}$ in terms of θ . [5 marks]

(d) The diagram below (not drawn to scale) shows the curve $y = x^2 + 3$ and the line $y = 4x$.



(i) Determine the coordinates of the points P and Q at which the curve and the line intersect. [4 marks]

(ii) Calculate the area of the shaded region. [5 marks]

Total 25 marks

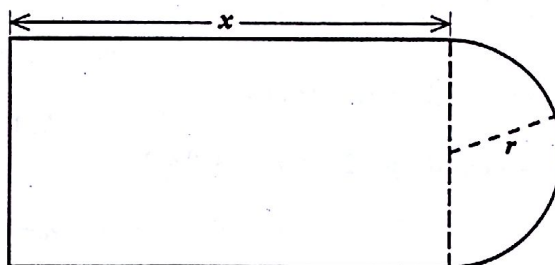
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6. (a) (i) By using the substitution $u = 1 - x$, find $\int x(1-x)^2 dx$. [5 marks]

(ii) Given that $f(t) = 2 \cos t$, $g(t) = 4 \sin 5t + 3 \cos t$,

show that $\int [f(t) + g(t)] dt = \int f(t) dt + \int g(t) dt$. [4 marks]

(b) A sports association is planning to construct a running track in the shape of a rectangle surmounted by a semicircle, as shown in the diagram below. The letter x represents the length of the rectangular section and r represents the radius of the semicircle.



The perimeter of the track must be 600 metres.

(i) Show that $r = \frac{600 - 2x}{2 + \pi}$. [2 marks]

(ii) Hence, determine the length, x , that maximises the area enclosed by the track. [6 marks]

(c) (i) Let $y = -x \sin x - 2 \cos x + Ax + B$, where A and B are constants.

Show that $y' = x \sin x$. [4 marks]

(ii) Hence, determine the specific solution of the differential equation

$$y'' = x \sin x,$$

given that when $x = 0$, $y = 1$ and when $x = \pi$, $y = 6$. [4 marks]

Total 25 marks

END OF TEST

FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.