Module 1: Reasoning and Logic Summary For Students

**Logical statements**

The study of logic is concerned with the truth or falsity of statements.

***Deﬁnition: A statement or proposition is a sentence which can be classiﬁed as true or false (but not both) without ambiguity. The truth or falsity of the statement is known as the truth value.***

Example 2 Consider the following sentences:

1. “Every function is diﬀerentiable” is a statement with truth value “false”.

2.”x<2” is true for some x and false for some others. If we have a particular context in mind, then it could be a statement. Otherwise it is not.

3. “27 is a prime number” is a statement with truth value false.

4. This sentence is false. ..

**Logical Connectives**

In studying mathematical logic we shall not be concerned with the truth value of any particular simple statement. What will be important is how the truth value of a compound statement is determined from the truth values of its simpler parts. To obtain such compound statements it is necessary to introduce the concept of a connective

***Deﬁnition: A sentential connective is a logic symbol representing an operator that combines statements into a new statement.***

Statements with connectives are called compound statements. Statements without connectives are known as atomic statements. The sentential connectives are “not”,”and”,”or”,”if ...then”, and ”if and only if ”. The respective operators for these connectives are negation, conjunction, disjunction, implication and equivalence respectively. We shall introduce each of them now

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Negation (NOT, ˜ )** | |  | **Conjunction (AND,^)** | | |  | **Disjunction (OR, v)** | | |
|  |  |  | p | q | p ^ q |  | p | q | p v q |
| p | ᷉ p (read not p) |  | T | T | T |  | T | T | T |
| T | F |  | F | T | F |  | F | T | T |
| F | T |  | T | F | F |  | T | F | T |
|  |  |  | F | F | F |  | F | F | F |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Implication,Conditional** ⇒  **(if …then….)** | | |  | **Biconditional** ⇔  **(if and only if)…iff** | | |
| Antecedent | Consequent |  |  |  |  |  |
| p | q | p⇒q |  | p | q | p ^ q |
| T | T | T |  | T | T | T |
| *T* | *F* | *F* |  | T | F | F |
| F | T | T |  | F | T | F |
| F | F | T |  | F | F | T |

• ***Implication***: A statement of the form If p then q is called an implication or a conditional statement. The if-statement p is called the antecedent (hypothesis) and the then-statement q is called the consequent (conclusion). ***The convention adopted for the truth value of the implication is that it will be called false only when the antecedent is true and the consequent is false***. If we denote the implication ”if p then q” by p⇒ q, we obtain the following truth table . (Some use → instead of ⇒)

In English there are several ways to express the same mathematical condition p ⇒ q. These are

”if p, then q ” “q provided that p” ”p implies q ” ” q whenever p” ”p only if q ” ”q is a necessary condition for p” ”q if p” ” p is a suﬃcient for condition q”

|  |  |
| --- | --- |
| ***Biconditional*** |  |

|  |  |
| --- | --- |
| Example 1:Examine the sentences below. | |
| |  |  | | --- | --- | | Given: | p: A polygon is a triangle. | | q: A polygon has exactly 3 sides. | | Problem: | Determine the truth values of this statement: (p⇒q) ^(q ⇒p) | |  |

|  |
| --- |
| The [compound statement](javascript:popUpWindow('compound_statement')) (p →q)^(q →p) is a conjunction of two [conditional statements](javascript:popUpWindow('conditional_statement')). In the first conditional, p is the hypothesis and q is the conclusion; in the second conditional, q is the hypothesis and p is the conclusion. Let's look at a truth table for this compound statement. |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| p | q | p⇒q | q⇒p | (p⇒q)^(q⇒p) |
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

In the truth table above, when p and q have the same truth values, the compound statement (p⇒q)^(q⇒p) is true. When we combine two conditional statements this way, we have a **biconditional**.

|  |  |
| --- | --- |
|  | |
| ***Definition::*** | ***A biconditional statement is defined to be true whenever both conditional statements have the same truth value. The biconditional operator is denoted by a double-headed arrow* ⇔*. The biconditional p* ⇔*q represents "p if and only if q," where p is a hypothesis and q is a conclusion.*(Some use ↔ instead of ⇔)** |
|  | |
|  |  |

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| *N.B. When proving the statement p iff q, it is equivalent to proving both of the statements "if p, then q" and "if q, then p." (In fact, this is exactly what we did in Example 1.) In each of the following examples, we will determine whether or not the given statement is biconditional using this method*. |

Equivalence:

***Definition: Two compound statements p and q are said to be logically equivalent if one is true if and only if the other is true. In other words, two compound propositions are logically equivalent whenever they display the same truth tables. In this case we write p≡ q.***

Example 2. (˜(˜p)) is logically equivalent to p Example 3. p ⇒ q is logically equivalent to [(˜p) ∨ q]

|  |  |  |
| --- | --- | --- |
| p | (˜p) | (˜(˜p)) |
| T | F | T |
| F | T | F |

|  |  |  |
| --- | --- | --- |
| p | q | p ⇒ q |
| T | T | T |
| F | T | T |
| T | F | F |
| F | F | T |

|  |  |  |
| --- | --- | --- |
| (˜p) | q | [(˜p) ∨ q] |
| F | T | T |
| T | T | T |
| F | F | F |
| T | F | T |

When two compound statements p and q are logically equivalent, their equiv-alence is a new statement with a truth value ”true” in all cases. Such a statement is known as a ***tautology.***

**Deﬁnition : A compound proposition is said to be a tautology if it is always true regardless of the truth value of the simple propositions from which it is constructed. It is a contradiction if it is always false. Hence a contradiction and a tautology are a negation of each other.**

Example 3. p ∨ (˜p) is a tautology, while p∧ (˜p) is a contradiction.

|  |  |  |
| --- | --- | --- |
| p | (˜p) | p ∨ (˜p) |
| F | T | T |
| T | F | T |

|  |  |  |
| --- | --- | --- |
| p | (˜p) | p∧ (˜p) |
| F | T | F |
| T | F | F |

Example 4. [p∧ (p⇒ q)] ⇒ q is a tautology

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| p | q | p ⇒ q | p∧ (p⇒ q) | [p∧ (p⇒ q)] ⇒ q |
| T | T | T | T | T |
| F | T | T | F | T |
| T | F | F | F | T |
| F | F | T | F | T |

**Inverse – To make the conditional statement negative (in other words the inverse), add NOT to the hypothesis and to the conclusion**

E.g. Consider the statement p: It is raining q: The sky is grey

Conditional p⇒ q Inverse ᷉ p⇒ ᷉ q

It is raining then the sky is grey. It is not raining then the sky is not grey

**Converse – To find the converse, the hypothesis and the conclusion are flipped, so that the hypothesis becomes the conclusion and the conclusion becomes the hypothesis.**

Conditional p⇒ q Converse q⇒ p

It is raining then the sky is grey. If the sky is grey then it is raining.

**Contrapositive – To find the contrapositive, make the converse statement negative.**

Conditional q⇒ p Contrapositive ᷉ q⇒ ᷉p

If the sky is grey then it is raining. If the sky is not grey then it is not raining.

***It also negates the hypothesis, and makes it the conclusion.***

Here are the truth tables for an implication, its contrapositive, and its converse:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| A | B | not A  ˜(A) | not B  ˜(B) | ***an implication:***  If A then B  (A⇒B) | ***The contrapositive of the***  ***implication***  If (not B) then (not A)  (˜B) ⇒ (˜A) | ***The converse of the implication***  If B then A  (B ⇒A) |
| T | T | F | F | T | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | F |
| F | F | T | T | T | T | T |

*An analysis of these truth tables shows the following: An implication is equivalent to its contrapositive.*

*Thus, the sentences ‘If A, then B’ and ‘If (not B), then (not A)’ are completely interchangeable:*

*if one is true, so is the other; if one is false, so is the other..*

*An implication is NOT equivalent to its converse.*

*Thus, the sentences ‘If A, then B’ and ‘If B, then A’ are not interchangeable.*

*The truth of each sentence must be investigated separately.*