02134020/CAPE/MS/SPEC

CARIBBEAN EXAMINATIONS COUNCIL

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 1

ALGEBRA, GEOMETRY AND CALCULUS

SPECIMEN PAPER

PAPER 02

SOLUTIONS AND MARK SCHEMES

SECTION A

(MODULE 1)

Question 1

(a)

p	**q**	**~p** (1 mark)	p → q (1 mark)	~p ∨ q (1 mark)
T	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**

T = true   F = false

[1 may be used for T and 0 for F]

[3 marks]

(ii) **p** → **q** and ~p ∨ q are logically equivalent since columns 4 and 5 are identical.

[2 marks]

(iii) p □ (p → q) = p □ (~p □ q) (1 mark)

= (**p** □ ~**p**) □ (**p** □ **p**)… distribute □ over □

= f □ (p □ ~ **q**) (1 mark)

= (**p** □ **q**) (1 mark)

[3 marks]

Specific objectives: (A) 2, 4

(b) (i) $x * y = x + y - 1$, $\forall\, x, y$ in $\mathbb{R}$

x, y □ $\mathbb{R}$ □ $x + y \in \mathbb{R}$ (sum of 2 real numbers) (1 mark)

□ $x + y - 1 \in \mathbb{R}$ (difference of 2 real numbers)

□ x □ $y \in \mathbb{R}$ (1 mark)

□ □ is closed in real numbers (1 mark)

[3 marks]

(ii) $x * y = x + y - 1 = y + x - 1$ (addition is commutative) (1 mark)

□ $y * x$ (1 mark)

□ $*$ is commutative in $\mathbb{R}$

[2 marks]

(b) (iii) $(x \;□\; y) \;□\; z = (x + y - 1) \;□\; z$ for $x, y, z \in \mathbb{R}$

$= (x + y - 1) + z - 1$
$= x + y + z - 1 - 1$
$= x + y + z - 2$ [2 marks]

$x \;□\; (y \;□\; z) = x \;□\; (y + z - 1)$
$= x + (y + z - 1) - 1$
$x + y + z - 2$ [1 mark]

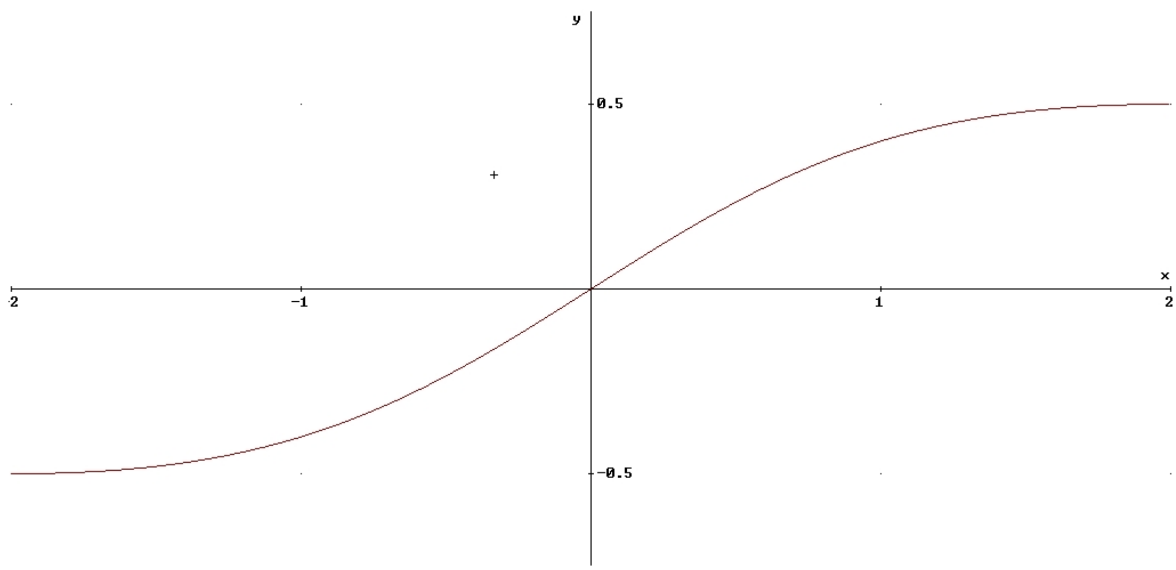
□ $(x \;□\; y) \;□\; z = x \;□\; (y \;□\; z)$ for all $x, y, z \in \mathbb{R}$
□ □ is associative in $\mathbb{R}$ [1 mark]

Specific Objectives (B) 2.

(c) (i) $y = \frac{2x}{x^2+4}$ □ $y(x^2 + 4) = 2x$ (1 mark)
□ $yx^2 - 2x + 4y = 0$

For x real, $(-2)^2 - 4y(4y) \geq 0$ (1 mark)
□ $4y^2 - 1 \leq 0$ (1 mark)
□ $(2y - 1)(2y + 1) \leq 0$ (1 mark)
□ $-\frac{1}{2} \leq y \leq \frac{1}{2}$ (1 mark)
[5 marks]

(ii) $|x| < 2$ □ $-2 \leq x \leq 2$



[3 marks]

Total 25 marks

Specific Objectives (C) 5; (F) 2.

<u>Question 2</u>

(a) Let $\mathrm{f}(x) = 2x^3 + px^2 + qx + 2$

(i) $\mathrm{f}(-1) = 0 \Rightarrow -2 + p - q + 2 = 0 \Rightarrow p = q$ (1 mark)

$\mathrm{f}\left(\frac{1}{2}\right) = 0 \Rightarrow \frac{1}{4} + \frac{p}{4} + \frac{q}{2} + 2 = 0 \Rightarrow p + 2q = -9$ (2 mark)

$\Rightarrow p = q = -3$ (1 mark)

(ii) $\mathrm{f}(x) = (2x-1)(x+1)(x-k)$ (1 mark)

$\equiv 2x^3 + px^2\ qx + 2$

$\Rightarrow k = 2$

$\Rightarrow$ the remaining root is 2 (1 mark)

Alternatively

(a) Let d be the third root of $\mathrm{f}(x) = 0$

(i) Then $-1 \times \frac{1}{2} \times d = -\frac{2}{2} \square \frac{d}{2} = 1 \square d = 2$ [3 marks]

$d = 2 \square -1 + \frac{1}{2} + 2 = -\frac{p}{2} \square p = -3$

And $-\frac{1}{2} - 2 + 1 = \frac{q}{2} \square q = -3$ [4 marks]

(ii) Let P_n be the statement $\sum_{r=1}^{n} (6r + 5) = n(3n + 8)$

For $n = 1$, L.H.S. of P_1, is $6 + 5 = 11$ and R.H.S. of P_1

$= 1(3 + 8)$

$= 11$ (1 mark)

So P_n is true for $n = 1$ (1 mark)

Assume that P_n is true for $n = k$, i.e

$11 + 17 + \ldots + (6k + 5) = k(3k + 8)$ (1 mark)

Then, we need to prove P_n is true for $n = k + 1$

Now $\sum_{r=1}^{k+1} (6r+5) = 11 + 17 + \ldots + (6k + 5) + [6(k + 1) + 5]$ (1 mark)

$= k(3k + 8) + [6(k + 1) + 5]$ (1 mark)

$= 3k^2 + 8k + 6k + 6 + 5$

$= 3k^2 + 14k + 11$ (1 mark)

$= (3k + 11)(k + 1)$ (1 mark)

$= (k + 1)[3(k + 1) + 8]$ (1 mark)

Thus, if P_n is true when $n = k$, it is also true with $n = (k + 1)$; (1 mark)

i.e. $\sum_{r=1}^{n} (6r + 5) = n(3n + 8)\ \forall n \in \mathbb{N}$. (1 mark)

[10 marks]

(c) $e^{2x} + 2e^{-2x} = 3$ □ $e^{2x} + \frac{2}{e^{2x}} = 3$ (1 mark)

□ $(e^{2x})^2 - 3e^{2x} + 2 = 0$ (1 mark)

□ $(e^{2x} - 2)(e^{2x} - 1) = 0$ (2 mark)

□ $e^{2x} = 2$ or $e^{2x} = 1$ (2 mark)

□ $2x = \ln 2$ or $2x = 0$

□ $x = \frac{1}{2}\ln 2$ or $x = 0$ (2 mark)

[8 marks]

Alternatively Let $y = e^{2x}$, giving $y^2 - 3y + 2 = 0$ etc.

Total 25 marks

Specific Objectives (B) 5; (D) 4,7; (G) 1, 2

SECTION B
(MODULE 2)

Question 3

(a) (i) LHS $\equiv \dfrac{\sin 3\theta + \sin\theta}{\cos 3\theta + \cos\theta}$ (1 mark)

$$\equiv \frac{2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{(3\theta-\theta)}{2}\right)}{2\cos\left(\frac{3\theta+\theta)}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right)}$$ (1 mark)

$$\equiv \frac{\sin 2\theta}{\cos 2\theta}$$ (1 mark)

$\equiv \tan 2\theta$ (1 mark)

$\equiv$ RHS

[4 marks]

(ii) $2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right) + \sin 2\theta = 0$ (1 mark)

$\Rightarrow\ 2\sin 2\theta\ \cos\theta + \sin 2\theta\ = 0$ (1 mark)

$\Rightarrow\ \sin 2\theta\,(2\cos\theta + 1) = 0$ (1 mark)

$\Rightarrow\ \sin 2\theta = 0$, that is $\theta = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$, or (3 marks)

$\cos\theta = \dfrac{-1}{2}$, that is $\theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ (1 mark)

[7 marks]

(b) (i) $r = \sqrt{(6^2 + 8^2)} = 10,\ x = \tan^{-1}\left(\frac{3}{4}\right) = 36.9^{\circ}$ (2 marks)

$\therefore\ f(\theta) = 10\cos(\theta - 36.9^{\circ})$ (1 mark)

[3 marks]

(ii) $g(\theta) = \dfrac{10}{10 + 10\cos(\theta - 36.9^{o})}$ (1 mark)

Minimum value of g (θ) occurs when denominator has maximum value i.e.
$\cos(\theta - 36.9) = 1$. (1 mark)

Min $g(\theta) = \dfrac{10}{10+10} = \dfrac{1}{2}$ occurs
(when $\theta - 36.9 = 0$), when $\theta = 36.9$. (2 marks)

[4 marks]

(c) (i) $\overrightarrow{\mathbf{BC}} = 2\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{\mathbf{BA}} = 2\mathbf{i} - 2\mathbf{k}$ [2 marks]

(ii) $\mathbf{n}.\overrightarrow{\mathbf{BC}} = (\mathbf{i} + \mathbf{j} + \mathbf{k}).(2\mathbf{j} - 2\mathbf{k}) = 0 + 2 - 2 = 0$ (1 mark)

$\mathbf{n}.\overrightarrow{\mathbf{BA}} = (\mathbf{i} + \mathbf{j} + \mathbf{k}).(2\mathbf{i} - 2\mathbf{k}) = 2 + 0 - 2 = 0$ (1 mark)

So **n** is perpendicular to the plane through A, B and C.

[2 marks]

(iii) Let $\mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}$ with $\mathbf{r}.\mathbf{n} = d$ (1 mark)

represent the plane through A, B and C.

At the point A, $\mathbf{r} = 2\mathbf{i}$ so $\mathbf{r}.\mathbf{n} = d$ (1 mark)

□ $(2\mathbf{i}).(\mathbf{i} + \mathbf{j} + \mathbf{k}) = d$ □ $d = 2$

Hence the Cartesian equation of the plane is $(x\mathbf{i} + y\mathbf{i} + z\mathbf{k}).(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2$ (1 mark)

□ $x + y + z = 2$

[3 marks]

[Total 25 marks]

Specific Objectives: (A) 3, 5, 7, 9, 10; (C) 7, 10

Question 4

(a) (i) L: $x + 2y = 7$ is a tangent to $x^2 + y^2 - 4x = 0$ if $x + 2y = 7$ if L touches the circle

at 2 coincident points (1 mark)

Now, $x = 7 - 2y$ (1 mark)

□ $(7 - 2y)^2 - 4(7 - 2y) + y^2 - 1 = 0$ (1 mark)

□ $y^2 - 4y + 4 = 0$ (1 mark)

□ $(y - 2)^2 = 0$ (1 mark)

□ $y = 2$(twice) (1 mark)

□ when $y = 2,\ x = 3$ (1 mark)

So L touches the circle at (3,2) (1 mark)

[8 marks]

(ii) a) Let $Q \equiv$ point diametrically opposite to (3, 2).

The centre of C is (2,0)

so $\frac{3+x}{2} = 2,\ x = 1$ (1 mark)

and, $\frac{2+y}{2} = 0,\ y = -2$ (1 mark)

$\therefore Q = (1,-2)$

Tangent M at Q: $y + 2 = \frac{-1}{2}(x - 1)$ (2 marks)

$2y + x + 3 = 0$ (1 mark)

[5 marks]

b) The equation of the diameter is $x + 2y = 2 + 0 = 2$ (1 mark)

This meets C where

$(2 - 2y)^2 + y^2 - 4(2 - 2y) - 1 = 0$ (1 mark)

□ $4 - y + 4y^2 + y^2 - 8 + y - 1 = 0$ (1 mark)

□ $5y^2 - 5 = 0$ (1 mark)

□ $y = \pm 1$

Coordinates of points of intersection are (1, 0), (–1, 4) (2 marks)

[6 marks]

(b) $x(1+t) = t \qquad y(1+t) = t^2$

$$\Rightarrow \quad \frac{y(1+t)}{x(1+t)} = \frac{t^2}{t}$$ (1 mark)

$$\Rightarrow \quad \frac{y}{x} = t$$ (1 mark)

$$\therefore \quad x = \frac{y/x}{1+y/x}$$ (1 mark)

$$\Rightarrow \quad x = \frac{y}{x+y}$$ (1 mark)

$$\Rightarrow \quad y = \frac{x^2}{1-x}$$ (2 marks)

[6 marks]

Total 25 marks

Specific Objectives: (B) 1, 2, 3, 4

SECTION C
(MODULE 3)

<u>Question 5</u>

(a) $\lim_{h \to 0} \dfrac{h}{\sqrt{x+h} - \sqrt{x}} = \lim_{h \to} \dfrac{h(\sqrt{x+h} + \sqrt{x})}{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}$ (1 mark)

$= \lim_{h \to 0} \dfrac{h(\sqrt{x+h} + \sqrt{x})}{(x+h) - x}$ (1 mark)

$= \dfrac{\lim}{h \to 0} \dfrac{h\left(\sqrt{x+h} + \sqrt{x}\right)}{h}$ (1 mark)

$= \lim_{h \to 0} \left(\sqrt{x+h} + \sqrt{x}\right)$ (1 mark)

$= \sqrt{x} + \sqrt{x}$ (1 mark)

$= 2\sqrt{x}$ (1 mark)

[6 marks]

(b) $f^{11}(x) = 18x + 4$

□ $f^{1}(x) = 9x^2 + 4x + c$ (1 mark)

□ $f(x) = 3x^3 + 2x^2 + cx + d$ (2 marks)

Now $f(2) = 14$ □ $32 + 2c + d = 14$

□ $2c + d = -18$…. (i) (1 mark)

$f(3) = 74$ □ $99 + 3c + d = 74$

□ $3c + d = -25$ …..(ii) (1 mark)

From (i) + (ii), $c = -7$ (1 mark)

$d = -4$ (1 mark)

So $f(4) = 3(4^3) + 2(4^2) - 7(4) - 4$

$= 192 + 32 - 32$

$= 192$ (1 mark)

[8 marks]

(c) (i) $y = \dfrac{x}{1+x^2}$

$$\frac{dy}{dx} = \frac{(1+x^2)1 - x(2x)}{(1+x^2)^2}$$ (2 marks)

$$= \frac{1-x^2}{(1+x^2)^2}$$ (1 mark)

$$= \frac{1}{(1+x^2)^2} - \frac{x^2}{(1+x^2)^2}$$

$$= \frac{1}{(1+x^2)^2} - \left(\frac{x}{1+x^2}\right)^2$$ (1 mark)

$$= \frac{1}{(1+x^2)^2} - y^2$$ (1 mark)

[5 marks]

(ii) $$\frac{d^2y}{dx^2} = \frac{(1+x^2)^2(-2x) - (1-x^2)2(1+x^2)(2x)}{(1+x^2)^4}$$ (3 marks)

$$= \frac{(1+x^2)(2x)\,[-(1+x^2) - 2 + 2x^2]}{(1+x^2)^4}$$ (2 marks)

$$= \frac{2y\,(x^2-3)}{(1+x^2)^2}$$ (1 mark)

[6 marks]

Total 25 marks

Specific Objectives: (A) 3, 4, 5, 6; (B) 8, 9, 16, (C) 1, 2, 3, 5, 9

Question 6

(a) (i) Finding the equation of the tangent PQ.

a) $\frac{dy}{dx} = 3x^2$ (1 mark)

$\left(\frac{dy}{dx}\right)_{x=3} = 3(3)^2 = 27$ (1 mark)

Equation of tangent:

$y - 27 = 27(x - 3)$ (1 mark)

$y = 27x - 54$ (1 mark)

[4 marks]

b) Q has coordinates (2, 0) (1 mark)

[1 mark]

(ii) a) Area = $\int_0^3 y\,dx - \frac{1}{2}(3-2)(27)$ (2 marks)

$= \frac{1}{4}x^4 \Big|_0^3 - \frac{27}{2}$ (2 marks)

$= \frac{81}{4} - \frac{27}{2}$

$= \frac{27}{4}$ units^2 (1 mark)

[5 marks]

Required Volume = $\int_0^3 \pi y^2\,dx$ – Volume of the cone with radius 27 units and height 1 unit. (1 mark)

$= \pi\int_0^3 x^6 dx - \frac{1}{3}\pi(27)^2$ (1 mark)

$= \pi\frac{x^7}{7}\Big|_0^3 - \frac{1}{3}\pi(3^6)$

(1 mark)

$= \pi\left(\frac{3^7}{7}\right) - \frac{1}{3}\pi(3^6)$

$= \frac{\pi}{7}\left(3^7 - 7(3^5)\right)$ (1 mark)

$= \frac{\pi}{7}\left(2\times 3^5\right)$ units^3 (1 mark)

[5 marks]

(b) (i) $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 8x + 5$

$y = x^3 - 4x^2 + 5x + \mathrm{C}$ (1 mark)

substituting; $y = 3$ at $x = 0$

$\mathrm{C} = 3$ (1 mark)

$y = x^3 - 4x^2 + 5x + 3$ (1 mark)

[3 marks]

(ii) $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ □ $3x^3 - 8x + 5 = 0$ (1 mark)

$(3x - 5)(x - 1) = 0$ (1 mark)

$x = \frac{5}{3}; 1$ (1 mark)

$y = \frac{131}{27}, 5$ (1 mark)

co-ordinates are $\left(\frac{5}{3}, \frac{131}{27}\right)$, $(1, 5)$

$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6x - 8$ (1 mark)

$\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_{x=5/3} > 0$ □ $\left(\frac{5}{3}, \frac{131}{27}\right)_{\max}$ (1 mark)

$\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_{x=1} < 0$ □ $(1, 5)_{\min}$ (1 mark)

[7 marks]

Total 25 marks

Specific Objective(s): (B) 11, 13, 14, 15, 16, 17;
(C) 8 (i), (ii)

END OF TEST