Proof by Induction

Introduction

Let us consider the following two expressions:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 \qquad \text{and} \qquad \frac{1}{4}n^2(n+1)^2.$$

Or in slightly more elegant notation

$$\sum_{k=1}^{n} k^3 \qquad \text{and} \qquad \frac{1}{4}n^2(n+1)^2.$$

The following table shows the results for the first few values of n

n	$\sum_{k=1}^{n} k^3$	$\frac{1}{4}n^2(n+1)^2$
1	1	1
2	9	9
3	36	36
4	100	100
5	225	225
6	441	441.

So we can see that it *appears* that the two terms are equal. However a Mathematician[1] will not be happy with this. How do we know that when n is 100,000 or 100,000,000 the two terms will still be equal? The answer is that we *don't*! But with a process known as *Proof by Induction* we can prove that the two terms will always be the same for all n.

Theory

Proof by Induction is a method of proving that two terms will always be equal for any value of n. The basic method is as follows

1. Demonstrate that the two terms are the same for some starting value of n. Usually one uses $n = 1$ or $n = 0$, but you can start at any n you fancy.
2. Assume the equality holds for $n = k$.
3. Try and show that *if* it is true for $n = k$ *then* it is also true for $n = k + 1$.

If it is true for some fixed value (STEP 1) and *if* it is true for $n = k$ *then* it is true for $n = k+1$ then we will have proved that it is true for all values of n greater than of equal to the one chosen in STEP 1. Think of the following diagrammatic representation:

$$\boxed{n = k \Rightarrow n = k+1} \text{ so } \boxed{n = 1 \Rightarrow n = 2 \Rightarrow n = 3 \Rightarrow n = 4 \Rightarrow n = 5 \Rightarrow n = 6 \Rightarrow n = 7 \ldots}$$

Let us try and use this technique to try and prove $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$; a very well known result from C2.

1. The statement is true for $n = 1$ because both sides of the statement are clearly 1.
2. Assume the equality holds for k. i.e. $1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k+1)$.

[1] We forget, for the moment (but alas not forever), about the inferior Engineer or Physicist.

3. We therefore *wish to show* that $1+2+3+\cdots+k+(k+1)=\frac{1}{2}(k+1)(k+2)$ (the formula we would expect for $k+1$). We therefore take

$$\begin{aligned} 1+2+3+\cdots+k+\underline{(k+1)} &= \frac{1}{2}k(k+1)+\underline{(k+1)} \\ &= \frac{1}{2}[k(k+1)+2(k+1)] \\ &= \frac{1}{2}(k+1)(k+2). \end{aligned}$$

   Which is what we wanted, so since the formula is true for $n=1$ and *if* it is true for $n=k$ *then* it is also true for $n=k+1$ we can conclude that the formula is true for all $n \geqslant 1$. And we are done.

Example

1. The example at the beginning; prove that $1^3+2^3+3^3+\cdots+n^3=\frac{1}{4}n^2(n+1)^2$.

   Well firstly the equation clearly holds for $n=1$ (both sides are 1).

   We assume it holds for $n=k$, so $1^3+2^3+3^3+\cdots+k^3=\frac{1}{4}k^2(k+1)^2$.

   We therefore need to show that $1^3+2^3+3^3+\cdots+k^3+(k+1)^3=\frac{1}{4}(k+1)^2(k+2)^2$.

   So the algebraic slog begins:

$$\begin{aligned} 1^3+2^3+3^3+\cdots+k^3+\underline{(k+1)^3} &= \frac{1}{4}k^2(k+1)^2+\underline{(k+1)^3} \\ &= \frac{1}{4}[k^2(k+1)^2+4(k+1)^3] \\ &= \frac{1}{4}[(k+1)^2(k^2+4(k+1))] \\ &= \frac{1}{4}[(k+1)^2(k^2+4k+4)] \\ &= \frac{1}{4}(k+1)^2(k+2)^2. \end{aligned}$$

   Which is what we wanted, so we are done. True for $n=1$. If true for $n=k$ then true for $n=k+1$, so true for all $n \geqslant 1$.