Worksheet 4.12 Induction

Mathematical Induction is a method of proof. We use this method to prove certain propositions involving positive integers. Mathematical Induction is based on a property of the natural numbers, $\mathbb{N}$, called the Well Ordering Principle which states that evey nonempty subset of positive integers has a least element.

There are two steps in the method:

Step 1: Prove the statement is true at the starting point (usually $n = 1$).

Step 2: Assume the statement is true for n.
Prove the statement is true for $n + 1$ (using the assumption).

Example 1 : Prove $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$ for all $n \in \mathbb{N}$

Step 1: [We want to show this is true at the starting point $n = 1$.]

$$\begin{aligned} \text{LHS} &= 1 \\ \text{RHS} &= 1^2 = 1 \end{aligned}$$

Since LHS=RHS, the statement is true for $n = 1$.

Step 2: Assume the statement is true for n.

i.e. $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$

[Want to show this is true for $n + 1$.

i.e. Want to show $1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2$]

$$\begin{aligned} \text{LHS} &= \underbrace{1 + 3 + 5 + \cdots + (2n - 1)}_{} + (2n + 1) \\ &= \qquad n^2 \qquad + (2n + 1) \quad \text{(by assumption)} \\ &= n^2 + 2n + 1 \\ &= (n + 1)^2 \\ &= \text{RHS} \end{aligned}$$

So, the statement is true for $n + 1$. Hence, the statement is true for all $n \in \mathbb{N}$, by induction. □

Example 2 : Prove $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

Step 1: [We want to show this is true at the starting point $n = 1$.]

$$\begin{aligned} \text{LHS} &= \sum_{k=1}^{n} k^2 = 1^2 = 1 \\ \text{RHS} &= \tfrac{1}{6}1(1+1)\,(2(1)+1) = 1 \end{aligned}$$

Since LHS=RHS, the statement is true for $n = 1$.

Step 2: Assume the statement is true for n.

i.e. $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

[Want to show this is true for $n + 1$.

i.e. Want to show $\sum_{k=1}^{n+1} k^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$]

$$\begin{aligned} \text{LHS} &= \sum_{k=1}^{n+1} k^2 \\ &= \underbrace{1^2 + 2^2 + \cdots + n^2} + (n+1)^2 \\ &= \tfrac{1}{6}n(n+1)(2n+1) + (n+1)^2 \quad \text{(by assumption)} \\ &= \tfrac{1}{6}(n+1)\,(n(2n+1) + 6(n+1)) \\ &= \tfrac{1}{6}(n+1)\,(2n^2 + 7n + 6) \\ &= \tfrac{1}{6}(n+1)(n+2)(2n+3) \\ &= \text{RHS} \end{aligned}$$

So, the statement is true for $n + 1$. Hence, the statement is true for all $n \in \mathbb{N}$, by induction. □

Example 3 : Prove $2^n > n^2$ for $n > 5$.

Step 1: [We want to show this is true at the starting point $n = 5$.]

$$\begin{aligned} \text{LHS} &= 2^5 = 32 \\ \text{RHS} &= 5^2 = 25 \end{aligned}$$

Since LHS > RHS, the statement is true for $n = 5$.

Step 2: Assume the statement is true for n i.e. $2^n > n^2$.

[Want to show this is true for $n+1$ i.e. want to show $2^{n+1} > (n+1)^2$]

$$\begin{aligned} \text{LHS} &= 2^{n+1} \\ &= 2^n \cdot 2 \\ &> 2n^2 \quad \text{(by assumption)} \\ &= n^2 + n^2 \\ &> n^2 + 2n + 1 \quad (\text{since } n^2 > 2n+1 \text{ for } n \geq 5) \\ &= (n+1)^2 \\ &= \text{RHS} \end{aligned}$$

So $2^{n+1} > (n+1)^2$ for $n \geq 5$ i.e. the statement is true for $n+1$ whenever $n \geq 5$. Hence, the statement is true for all $n \geq 5$, by induction. □

Example 4 : Prove that $9^n - 2^n$ is divisible by 7 for all $n \in \mathbb{N}$

Step 1: [We want to show this is true at the starting point $n = 1$.]

When $n = 1$, we have $9^1 - 7^1 = 7$ which is divisible by 7.

The statement is true for $n = 1$.

Step 2: Assume the statement is true for n.

i.e. Assume $9^n - 2^n$ is divisible by 7.

i.e. Assume $9^n - 2^n = 7m$ for some $m \in \mathbb{Z}$.

[Want to show this is true for $n+1$.

i.e. Want to show $9^{n+1} - 2^{n+1}$ is divisible by 7.]

$$\begin{aligned} 9^{n+1} - 2^{n+1} &= 9 \cdot 9^n - 2 \cdot 2^n \\ &= 9(7m + 2^n) - 2 \cdot 2^n \quad \text{(by assumption)} \\ &= 7(9m) + 9 \cdot 2^n - 2 \cdot 2^n \\ &= 7(9m) + 7 \cdot 2^n \\ &= 7(9m + 2^n), \end{aligned}$$

which is divisible by 7. So the statement is true for $n+1$. Hence, the statement is true for all $n \in \mathbb{N}$, by induction. □

Exercises:

1. Prove the following propositions for all positive integers n.

   (a) $1 + 5 + 9 + 13 + \cdots + (4n - 3) = \frac{1}{2}n(4n - 2)$

   (b) $\sum_{k=1}^{n} k = \frac{1}{2}n(n + 1)$

   (c) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2$

   (d) $10^1 + 10^2 + 10^3 + \cdots + 10^n = \frac{10}{9}(10^n - 1)$

   (e) $\sum_{r=1}^{n} r(r + 1) = \dfrac{n(n + 1)(n + 2)}{3}$

   (f) $\sum_{k=1}^{n} \dfrac{1}{(3k - 2)(3k - 1)} = \dfrac{n}{3n + 1}$

2. Prove the following by induction.

   (a) $2^n \geq 1 + n$ for $n \geq 1$

   (b) $3^n < (n + 1)!$ for $n \geq 4$

3. Prove that $8^n - 3^n$ is divisible by 5 for all $n \in \mathbb{N}$.

4. Prove that $n^3 + 2n$ is divisible by 3 for all $n \in \mathbb{N}$.

5. Prove by induction that, if p is any real number satisfying $p > -1$, then $(1+p)^n \geq 1+np$ for all $n \in \mathbb{N}$.

6. Use induction to show that the nth derivative of x^{-1} is $\dfrac{(-1)^n n!}{x^{n+1}}$.