Worksheet 4.12  Induction

Mathematical Induction is a method of proof. We use this method to prove certain propositions involving positive integers. Mathematical Induction is based on a property of the natural numbers, \( \mathbb{N} \), called the Well Ordering Principle which states that every nonempty subset of positive integers has a least element.

There are two steps in the method:

**Step 1:** Prove the statement is true at the starting point (usually \( n = 1 \)).

**Step 2:** Assume the statement is true for \( n \).
Prove the statement is true for \( n + 1 \) (using the assumption).

**Example 1:** Prove \( 1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2 \) for all \( n \in \mathbb{N} \)

**Step 1:** [We want to show this is true at the starting point \( n = 1 \).]

\[
\text{LHS} = 1 \\
\text{RHS} = 1^2 = 1
\]

Since LHS = RHS, the statement is true for \( n = 1 \).

**Step 2:** Assume the statement is true for \( n \).
\[\text{i.e. } 1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2\]
[Want to show this is true for \( n + 1 \).
\[\text{i.e. } \text{Want to show } 1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2\]

\[
\text{LHS} = 1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) \\
= \underbrace{n^2 + (2n + 1)}_{\text{by assumption}} \\
= n^2 + 2n + 1 \\
= (n + 1)^2 \\
= \text{RHS}
\]

So, the statement is true for \( n + 1 \). Hence, the statement is true for all \( n \in \mathbb{N} \), by induction.
Example 2: Prove \( \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n + 1)(2n + 1) \) for all \( n \in \mathbb{N} \).

Step 1: We want to show this is true at the starting point \( n = 1 \).

\[
\begin{align*}
\text{LHS} &= \sum_{k=1}^{n} k^2 = 1^2 = 1 \\
\text{RHS} &= \frac{1}{6}(1 + 1)(2(1) + 1) = 1
\end{align*}
\]

Since LHS=RHS, the statement is true for \( n = 1 \).

Step 2: Assume the statement is true for \( n \).

i.e. \( \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n + 1)(2n + 1) \).

[Want to show this is true for \( n + 1 \).

i.e. Want to show \( \sum_{k=1}^{n+1} k^2 = \frac{1}{6}(n + 1)(n + 2)(2n + 3) \)]

\[
\begin{align*}
\text{LHS} &= \sum_{k=1}^{n+1} k^2 \\
&= 1^2 + 2^2 + \cdots + n^2 + (n + 1)^2 \\
&= \frac{1}{6}n(n + 1)(2n + 1) + (n + 1)^2 \quad \text{(by assumption)} \\
&= \frac{1}{6}(n + 1)(n(2n + 1) + 6(n + 1)) \\
&= \frac{1}{6}(n + 1)(2n^2 + 7n + 6) \\
&= \frac{1}{6}(n + 1)(n + 2)(2n + 3) \\
&= \text{RHS}
\end{align*}
\]

So, the statement is true for \( n + 1 \). Hence, the statement is true for all \( n \in \mathbb{N} \), by induction. \( \square \)

Example 3: Prove \( 2^n > n^2 \) for \( n > 5 \).

Step 1: We want to show this is true at the starting point \( n = 5 \).

\[
\begin{align*}
\text{LHS} &= 2^5 = 32 \\
\text{RHS} &= 5^2 = 25
\end{align*}
\]
Since LHS > RHS, the statement is true for $n = 5$.

**Step 2:** Assume the statement is true for $n$ i.e. $2^n > n^2$.

[Want to show this is true for $n + 1$ i.e. want to show $2^{n+1} > (n + 1)^2$]

\[
\text{LHS} \quad = \quad 2^{n+1} \\
\quad = \quad 2^n \cdot 2 \\
\quad > \quad 2n^2 \quad \text{(by assumption)} \\
\quad = \quad n^2 + n^2 \\
\quad > \quad n^2 + 2n + 1 \quad \text{(since $n^2 > 2n + 1$ for $n \geq 5$)} \\
\quad = \quad (n + 1)^2 \\
\quad = \quad \text{RHS}
\]

So $2^{n+1} > (n + 1)^2$ for $n \geq 5$ i.e. the statement is true for $n + 1$ whenever $n \geq 5$.
Hence, the statement is true for all $n \geq 5$, by induction. \( \square \)

**Example 4:** Prove that $9^n - 2^n$ is divisible by 7 for all $n \in \mathbb{N}$

**Step 1:** [We want to show this is true at the starting point $n = 1$.]

When $n = 1$, we have $9^1 - 7^1 = 7$ which is divisible by 7.

The statement is true for $n = 1$.

**Step 2:** Assume the statement is true for $n$.

i.e. Assume $9^n - 2^n$ is divisible by 7.

i.e. Assume $9^n - 2^n = 7m$ for some $m \in \mathbb{Z}$.

[Want to show this is true for $n + 1$.

i.e. Want to show $9^{n+1} - 2^{n+1}$ is divisible by 7.]

\[
9^{n+1} - 2^{n+1} \quad = \quad 9 \cdot 9^n - 2 \cdot 2^n \\
\quad = \quad 9(7m + 2^n) - 2 \cdot 2^n \quad \text{(by assumption)} \\
\quad = \quad 7(9m) + 9 \cdot 2^n - 2 \cdot 2^n \\
\quad = \quad 7(9m) + 7 \cdot 2^n \\
\quad = \quad 7(9m + 2^n),
\]

which is divisible by 7. So the statement is true for $n + 1$. Hence, the statement is true for all $n \in \mathbb{N}$, by induction. \( \square \)
Exercises:

1. Prove the following propositions for all positive integers $n$.

   (a) $1 + 5 + 9 + 13 + \cdots + (4n - 3) = \frac{1}{2}n(4n - 2)$

   (b) $\sum_{k=1}^{n} k = \frac{1}{2}n(n + 1)$

   (c) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2$

   (d) $10^1 + 10^2 + 10^3 + \cdots + 10^n = \frac{10}{9}(10^n - 1)$

   (e) $\sum_{r=1}^{n} r(r + 1) = \frac{n(n + 1)(n + 2)}{3}$

   (f) $\sum_{k=1}^{n} \frac{1}{(3k - 2)(3k - 1)} = \frac{n}{3n + 1}$

2. Prove the following by induction.

   (a) $2^n \geq 1 + n$ for $n \geq 1$

   (b) $3^n < (n + 1)!$ for $n \geq 4$

3. Prove that $8^n - 3^n$ is divisible by 5 for all $n \in \mathbb{N}$.

4. Prove that $n^3 + 2n$ is divisible by 3 for all $n \in \mathbb{N}$.

5. Prove by induction that, if $p$ is any real number satisfying $p > -1$, then $(1 + p)^n \geq 1 + np$ for all $n \in \mathbb{N}$.

6. Use induction to show that the $n$th derivative of $x^{-1}$ is $\frac{(-1)^n n!}{x^{n+1}}$. 